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Thermodynamics of the degenerate supersymmetric t–J model in one dimension

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Abstract. The one-dimensional SU(N)-invariant t–J model consists of electrons with N spin components on a lattice with nearest-neighbour hopping and spin exchange J. The multiple occupancy of the lattice sites is excluded. The model is integrable at the supersymmetric point, $t = J$. The discrete Bethe *ansatz* equations are analysed and the solutions are classified according to the string hypothesis. The thermodynamic Bethe *ansatz* equations are derived for arbitrary band filling in terms of thermodynamic energy potentials for the classes of eigenstates of the Hamiltonian. These equations are solved in limiting cases, e.g., $S = 1/2$, the ground state and the high-temperature limit. If the charge fluctuations are suppressed the Bethe *ansatz* equations map onto those of the SU(N)-invariant Heisenberg chain.

1. Introduction

The two-dimensional Hubbard model is believed to have the main features to explain many of the fundamental properties of the cuprate high-temperature superconductors [1]. The key ingredient is the motion of highly correlated electrons within the CuO planes. In the limit of very large on-site Coulomb repulsion the Hubbard model can be mapped onto the t–J model, for which numerous properties have been studied with approximate methods [2, 3]. It has been conjectured [1, 4] that the 1D and 2D variants of the Hubbard and t–J models have properties in common. Exact results in 1D are often more accessible than 2D ones and may provide a testing ground for approaches intended for more complex problems.

The one-dimensional t–J model for spin 1/2 was found to be integrable at the supersymmetric point by Sutherland [5]. This supersymmetry is related to the SU(3)-invariant Heisenberg chain of spin 1. The Bethe *ansatz* equations, the classification of states and the thermodynamic equations for this model were presented in [6]. These results were used by Bares and Blatter [7] to calculate the spectrum of elementary excitations explicitly, and by Kawakami and Yang [8] to obtain the exponents for the long-distance asymptotic of the correlation functions. In [9] we constructed an asymptotically exact solution in the low-electron-density limit for coupling parameters deviating from SU(3) symmetry, i.e. $t \neq J$, and Lee and Schlottmann [10] extended the Bethe *ansatz* solution at the supersymmetric point to an arbitrary number of spin components N.

In this paper we present the thermodynamic Bethe *ansatz* equations for the 1D supersymmetric t–J model with N spin components ($N = 2S + 1$), generalizing in this way the results of [10] to finite temperatures. The model under consideration is the following

$$H = -\sum_{is} P\left(c_{is}^{+}c_{i+1s} + c_{i+1s}^{+}c_{is}\right)P + V\sum_{iss'} n_{is}n_{i+1s'} + J\sum_{iss'} c_{is}^{+}c_{is'}c_{i+1s'}^{+}c_{i+1s} \tag{1.1}$$

where the hopping matrix element t has been equated to 1. Here c_{is} annihilates an electron at site i with spin component s, where $|s| \leqslant S$, n_{is} is the number operator, P is a projector excluding the *multiple occupancy* of every site, V is a charge interaction independent of the spin and J is a spin-exchange interaction. The generalized spin S can be thought of as composed of spin and orbital degrees of freedom.

Several special cases of this model are worth noting. (i) If $J = 0$ the spin components do not play a relevant role, and we only have to distinguish occupied and empty sites. The model then reduces to the spin 1/2 anisotropic XXZ Heisenberg chain. (ii) If there is one electron per site there are no charge dynamics and the system is just the Heisenberg chain of spin S with SU($2S+1$) invariance [5]. In addition, model (1.1) is integrable for arbitrary band filling in the following cases: (iii) $J = -V = \pm 1$, and (iv) $J = V = \pm 1$. These situations are related to the SU($2S+2$) invariance of the model with ($2S+1$) spin and one charge degree of freedom. The situation $J = -V = 1$ corresponds to the supersymmetric limit of the traditional t–J model (the t–J model represents the high-correlation limit of the degenerate Hubbard model only if $J \ll t$). The thermodynamics of this case for arbitrary spin is the subject of this paper.

The rest of the paper is organized as follows. In section 2 we restate the discrete Bethe *ansatz* equations derived previously in [10]. The structure of the ground and excited states is discussed and integral equations relating the densities of these states are given. Furthermore sets of thermodynamic equations are obtained by minimization of the free energy functional. In section 3 we recover as special cases the ground-state equations, the high-temperature limit and the Heisenberg model. Conclusions follow in section 4.

2. The thermodynamic equations

2.1. Bethe ansatz equations

We first consider the Hamiltonian (1.1) for $J = -V = 1$ for two particles in a box. The total wavefunction can be written as a product of a spin wavefunction and a coordinate wavefunction. Since the total wavefunction has to be antisymmetric, one of the two factors has to be symmetric and the other one antisymmetric. Only pairs of fermions forming an antisymmetric spin wavefunction are scattered [10], but not electrons in a symmetric spin state. The corresponding scattering matrix is given by

$$\mathsf{S}(k_1, k_2) = \frac{p_1 - p_2}{p_1 - p_2 + i}\,\mathsf{I} + \frac{i}{p_1 - p_2 + i}\,\mathsf{P} \tag{2.1}$$

where I is the identity matrix and P permutes the spin indices. Here p is related to the wavenumber k by $p = \frac{1}{2}\cot(k/2)$. It is easy to verify that (2.1) satisfies the triangular Yang–Baxter relation [11] and that a multiparticle scattering matrix can be written as a product of two-particle scattering matrices, (2.1).

The exact solution of the model can now be obtained by a standard procedure [10–12]. Imposing periodic boundary conditions, the N_e-particle problem reduces to the simultaneous solution of N_e eigenvalue equations. This eigenvalue problem has been solved by Sutherland [12] for an arbitrary Young tableau by means of a sequence of $(N-1)$ nested Bethe *ansatze*. Each Bethe *ansatz* leads to a new eigenvalue problem with the number of spin components reduced by one and gives rise to a set of rapidities. This procedure is applied successively until all internal degrees of freedom are eliminated.

Hence, within the framework of the Bethe *ansatz* each internal degree of freedom gives rise to a set of rapidities $\{\xi_\alpha^{(l)}\}$. For an SU(N)-invariant model there are then N such sets, $l = 0, \dots, N-1$, where the set for $l = 0$ corresponds to the charge rapidities, i.e., it is related to the wavenumbers $\{k_\alpha\}$ of the particles $\xi_\alpha^{(0)} = p_\alpha = \frac{1}{2}\cot(k/2)$. All rapidities within a given set have to be different. This latter property leads to Fermi statistics for rapidities associated with spin waves, which have an integer spin and are actually hard-core bosons. The rapidities are not independent of each other but coupled by the discrete Bethe *ansatz* equations [10]

$$\left(\frac{\xi_\alpha^{(0)} - \frac{i}{2}}{\xi_\alpha^{(0)} + \frac{i}{2}}\right)^{N_a} = \prod_{\beta=1}^{M_1} \frac{\xi_\alpha^{(0)} - \xi_\beta^{(1)} - \frac{i}{2}}{\xi_\alpha^{(0)} - \xi_\beta^{(1)} + \frac{i}{2}} \qquad \alpha = 1, \dots, M_0 \tag{2.2a}$$

$$\prod_{\beta=1}^{M_l} \frac{\xi_\alpha^{(l)} - \xi_\beta^{(l)} - i}{\xi_\alpha^{(l)} - \xi_\beta^{(l)} + i} = -\prod_{\beta=1}^{M_{l-1}} \frac{\xi_\alpha^{(l)} - \xi_\beta^{(l-1)} - \frac{i}{2}}{\xi_\alpha^{(l)} - \xi_\beta^{(l-1)} + \frac{i}{2}} \prod_{\beta=1}^{M_{l+1}} \frac{\xi_\alpha^{(l)} - \xi_\beta^{(l+1)} - \frac{i}{2}}{\xi_\alpha^{(l)} - \xi_\beta^{(l+1)} + \frac{i}{2}} \tag{2.2b}$$

$$l = 1, \dots, N-1 \qquad M_0 \equiv N_e \qquad M_N \equiv 0 \qquad \alpha = 1, \dots, M_l$$

where N_a is the number of sites in the chain, N_e is the number of electrons and M_l is the number of rapidities in the set $\{\xi_\alpha^{(l)}\}$. If n_{S-m} denotes the number of electrons with spin component m and $M_{i+1} = M_i - n_i$, then necessarily $N_e \equiv M_0 \geqslant M_1 \geqslant M_2 \geqslant \dots \geqslant M_{N-1} \geqslant M_N \equiv 0$. This solution corresponds to the Young tableau $(M_0 - M_1, M_1 - M_2, \dots, M_{N-2} - M_{N-1}, M_{N-1} - M_N)$. The energy eigenvalues of the Hamiltonian (1.1) and the magnetization are given by

$$E = -2N_e + 2\sum_{\alpha=1}^{M_0} \frac{1/2}{(\xi_\alpha^0)^2 + 1/4} \tag{2.3a}$$

$$S_z = \frac{1}{2}(N-1)N_e - \sum_{l=1}^{N-1} M_l. \tag{2.3b}$$

2.2. *Excitations*

The ground state and the excitations of the system are given by the self-consistent solutions of equations (2.2). The rapidities have in general complex values and in the thermodynamic limit (large N_a, N_e and M_l), they can be classified according to:

(i) real charge rapidities, belonging to the set $\{\xi_\alpha^{(0)}\}$, which correspond to unpaired propagating electrons;

(ii) complex spin and charge rapidities, which correspond to bound states of electrons with different spin components; and

(iii) strings of complex spin rapidities, which correspond to bound spin states.

Since only electrons with different spin components are scattered, i.e. experience an effective attractive interaction, we may build spin complexes of up to $(2S+1)$ electrons. A complex of n electrons ($n \leqslant 2S+1$) is characterized by one real $\xi^{(n-1)}$ rapidity and in general complex $\xi^{(l)}$ rapidities, $l < n-1$, given by

$$\xi_p^{(l)} = \xi^{(n-1)} + \frac{\mathrm{i}}{2}p \qquad l \leqslant n-1 \leqslant 2S$$
$$p = -(n-l-1), -(n-l-3), \ldots, (n-l-1). \tag{2.4}$$

These spin and charge strings form the classes (i) and (ii), which are already present in the ground state [10]. In class (iii) there is a set of strings of complex spin rapidities for each set of real spin rapidities $\{\xi_\alpha^{(l)}\}$, $l = 1, \ldots, 2S$. A string of length n is given by

$$\xi_{\alpha n}^{(l)\mu} = \Lambda_{\alpha n}^{(l)} + \frac{\mathrm{i}}{2}\mu \qquad \mu = -(n-1), -(n-3), \ldots, (n-1) \tag{2.5}$$

where $\Lambda_{\alpha n}^{(l)}$ is a real parameter and α is the running index in each set.

The above rapidities are inserted into equations (2.2) and the resulting coupled equations for the real $\{\xi_\alpha^{(l)}\}$ and $\{\Lambda_{\alpha n}^{(l)}\}$ are logarithmized. This generates a set of integer quantum numbers for each set of rapidities. In the thermodynamic limit we define the usual distribution functions for the rapidities: $\rho^{(l)}(\xi)$ for the real $\xi_\alpha^{(l)}$ and $\sigma_n^{(l)}(\Lambda)$ for the $\Lambda_{\alpha n}^{(l)}$, and similarly for the 'hole' distribution functions $\rho_{\mathrm{h}}^{(l)}(\xi)$ and $\sigma_{n\mathrm{h}}^{(l)}(\Lambda)$. 'Particle' and 'hole' densities are not independent in view of the Fermi statistics of the rapidities, but coupled by sets of linear integral equations. Fourier transforming the equations, we have

$$\begin{aligned}\hat{\rho}_{\mathrm{h}}^{(l)}(\omega) + \hat{\rho}^{(l)}(\omega) + \sum_{q=0}^{2S} \hat{\rho}^{(q)}(\omega) \exp\left(-\frac{|\omega|}{2}(l+q-p_{l,q})\right) \frac{\sinh[\frac{\omega}{2}(p_{l,q}+1)]}{\sinh(\frac{\omega}{2})} \\ + \sum_{n=1}^{\infty} \hat{\sigma}_n^{(l+1)}(\omega) \exp\left(-n\frac{|\omega|}{2}\right) = \exp\left(-(l+1)\frac{|\omega|}{2}\right)\end{aligned} \tag{2.6}$$

$$\begin{aligned}\hat{\sigma}_{m\mathrm{h}}^{(l)}(\omega) = \hat{\rho}^{(l-1)}(\omega) \exp\left(-m\frac{|\omega|}{2}\right) \\ + \sum_{n=1}^{\infty} \left[\hat{\sigma}_n^{(l-1)}(\omega) + \hat{\sigma}_n^{(l+1)}(\omega) - 2\cosh\left(\frac{\omega}{2}\right)\hat{\sigma}_n^{(l)}(\omega)\right] \\ \times \exp\left(-\frac{|\omega|}{2}\max(m,n)\right) \frac{\sinh(\frac{\omega}{2}\min(m,n))}{\sinh(\frac{\omega}{2})}.\end{aligned} \tag{2.7}$$

The last set of equations holds for $m = 1, \ldots, \infty$ with $\sigma_m^{(0)}$, $\sigma_{m\mathrm{h}}^{(0)}$, $\sigma_m^{(N)}$, and $\sigma_{m\mathrm{h}}^{(N)}$ being identically zero, and $p_{l,q} = \min(l,q) - \delta_{l,q}$. Here the caret denotes a Fourier transform. Equations (2.7) are equivalent to the following set

$$
\begin{aligned}
2\cosh\left(\frac{\omega}{2}\right)\hat{\sigma}^{(l)}_{m\mathrm{h}}(\omega) - \hat{\sigma}^{(l)}_{m+1\mathrm{h}}(\omega) - \hat{\sigma}^{(l)}_{m-1\mathrm{h}}(\omega) &= \hat{\sigma}^{(l+1)}_{m}(\omega) \\
&\quad + \hat{\sigma}^{(l-1)}_{m}(\omega) - 2\cosh\left(\frac{\omega}{2}\right)\hat{\sigma}^{(l)}_{m}(\omega) \qquad m \geqslant 2 \\
2\cosh\left(\frac{\omega}{2}\right)\hat{\sigma}^{(l)}_{1\mathrm{h}}(\omega) - \hat{\sigma}^{(l)}_{2\mathrm{h}}(\omega) - \hat{\rho}^{(l-1)}(\omega) &= \hat{\sigma}^{(l+1)}_{1}(\omega) \\
&\quad + \hat{\sigma}^{(l-1)}_{1}(\omega) - 2\cosh\left(\frac{\omega}{2}\right)\hat{\sigma}^{(l)}_{1}(\omega).
\end{aligned}
\tag{2.8}
$$

These equations differ only by their driving terms (independent terms) from the corresponding ones for the N-component one-dimensional fermion gas with attractive δ-function interaction and the degenerate Anderson impurity in the $U \to \infty$ limit [13].

2.3. Minimization of the free energy

The distribution functions $\rho^{(l)}$ and $\sigma^{(l)}_m$ are actually determined by minimizing the free energy

$$F = E - TS \tag{2.9}$$

where

$$E = -2N_{\mathrm{e}} + 2N_{\mathrm{a}} \sum_{m=0}^{2S} \int \mathrm{d}\xi \; \rho^{(m)}(\xi) \frac{\frac{1}{2}(m+1)}{\xi^2 + \frac{1}{4}(m+1)^2}. \tag{2.10}$$

T is the temperature and S is the sum of the distribution entropies of the rapidities, which, e.g., for $\rho^{(l)}(\xi)$ is given by (Fermi distribution since all the rapidities within one set must be different)

$$S(\rho^{(l)}) = \int \mathrm{d}\xi \left[\left(\rho^{(l)} + \rho^{(l)}_{\mathrm{h}}\right)\ln\left(\rho^{(l)} + \rho^{(l)}_{\mathrm{h}}\right) - \rho^{(l)}\ln\rho^{(l)} - \rho^{(l)}_{\mathrm{h}}\ln\rho^{(l)}_{\mathrm{h}}\right]. \tag{2.11}$$

The minimization of the free energy functional must be carried out considering the relations (2.6) and (2.7) and under the constraint of a constant number of particles for each spin component, n_l. The numbers n_l are given by

$$n_l = \sum_{q=2S-l}^{2S} \int \mathrm{d}\xi \; \rho^{(q)}(\Lambda) + \sum_{n=1}^{\infty} \int \mathrm{d}\Lambda \left(\sigma^{(2S-l)}_n(\Lambda) - \sigma^{(2S+1-l)}_n(\Lambda)\right) \tag{2.12}$$

and the total number of electrons becomes

$$N_{\mathrm{e}} = \sum_{l=0}^{2S} n_l = \sum_{l=0}^{2S} (l+1) \int \mathrm{d}\xi \; \rho^{(l)}(\xi). \tag{2.13}$$

The Lagrange multipliers corresponding to the conservation of n_l are denoted by A_l and represent the chemical potential (Fermi energy), the magnetic field, crystalline field splittings, etc. It is useful to define an energy potential for each class of excitations

$$\rho^{(l)}_{\mathrm{h}}/\rho^{(l)} = \exp(\epsilon_l/T) \qquad \sigma^{(l)}_{n\mathrm{h}}/\sigma^{(l)}_n = \exp(\varphi^{(l)}_n/T) = \eta^{(l)}_n. \tag{2.14}$$

From equations (2.6) and (2.7) we have that only one-half of the density functions are actually independent. There are many equivalent ways to minimize the free energy, depending on which of the functions are chosen to be independent. Below we present three different, though equivalent, sets of thermodynamic Bethe *ansatz* equations for the energy potentials $\epsilon_l(\xi)$ and $\varphi^{(l)}_n(\Lambda)$.

2.4. Thermodynamic Bethe ansatz equations

2.4.1. If all the $\rho^{(l)}(\xi)$ and the $\sigma_n^{(l)}(\Lambda)$ are independent we obtain

$$\begin{aligned}\epsilon_m(\xi) = {} & 2\frac{\frac{1}{2}(m+1)}{\xi^2+\frac{1}{4}(m+1)^2} - 2(m+1) - \sum_{q=2S-m}^{2S} A_q \\ & + T\sum_{q=0}^{2S}\int \mathrm{d}\xi' \ln\left(1+\exp\left(\frac{-\epsilon_q}{T}\right)\right) \\ & \times \int \frac{\mathrm{d}\omega}{2\pi}\exp\left(\mathrm{i}(\xi-\xi')\omega - \frac{|\omega|}{2}(m+q-p_{m,q})\right)\frac{\sinh[\frac{\omega}{2}(p_{m,q}+1)]}{\sinh(\frac{\omega}{2})} \\ & - T\sum_{n=1}^{\infty}\int \mathrm{d}\Lambda' \ln\left(1+(\eta_n^{(m+1)})^{-1}\right)\frac{1}{\pi}\frac{n/2}{(\xi-\Lambda')^2+n^2/4}\end{aligned} \qquad (2.15)$$

$$\begin{aligned}T\ln\left(1+\eta_n^{(m)}\right) = {} & n\,(A_{2S+1-m} - A_{2S-m}) + T\sum_{n'=1}^{\infty}\int \mathrm{d}\Lambda' \int\frac{\mathrm{d}\omega}{2\pi}\Big\{2\cosh\left(\frac{\omega}{2}\right) \\ & \times \ln\left(1+(\eta_{n'}^{(m)})^{-1}\right) - \ln\left[\left(1+(\eta_{n'}^{(m+1)})^{-1}\right)\left(1+(\eta_{n'}^{(m-1)})^{-1}\right)\right]\Big\} \\ & \times \exp\left(\mathrm{i}(\Lambda-\Lambda')\omega - \frac{|\omega|}{2}\max(n,n')\right)\frac{\sinh[\frac{\omega}{2}\min(n,n')]}{\sinh(\frac{\omega}{2})} \\ & + T\int \mathrm{d}\xi\,\ln(1+\exp(-\epsilon_{m-1}/T))\,\frac{1}{\pi}\frac{n/2}{(\xi-\Lambda)^2+n^2/4}.\end{aligned} \qquad (2.16)$$

2.4.2. On the other hand, if all the $\rho^{(l)}(\xi)$ and the $\sigma_{nh}^{(l)}(\Lambda)$ are independent then instead of equations (2.16) we obtain

$$\begin{aligned}\varphi_n^{(m)}(\Lambda) = {} & \delta_{n,1}\, T\int \mathrm{d}\xi\, G_0(\Lambda-\xi)\,\ln(1+\exp(-\epsilon_{m-1}/T)) \\ & + T\int \mathrm{d}\Lambda'\, G_0(\Lambda-\Lambda')\Big\{\ln\left[(1+\eta_{n+1}^{(m)})(1+\eta_{n-1}^{(m)})\right] \\ & - \ln\left[(1+(\eta_n^{(m+1)})^{-1})(1+(\eta_n^{(m-1)})^{-1})\right]\Big\}\end{aligned} \qquad (2.17)$$

or equivalently

$$\begin{aligned}\ln(1+(\eta_n^{(m)})^{-1}) = {} & \sum_{q=1}^{2S}\int \mathrm{d}\Lambda'\,\ln(1+\eta_n^{(q)})\int\frac{\mathrm{d}\omega}{2\pi}\mathrm{e}^{\mathrm{i}(\Lambda-\Lambda')\omega}\, 2\cosh\left(\frac{\omega}{2}\right)\hat{G}_{m,q}(\omega) \\ & - \sum_{q=1}^{2S}\int \mathrm{d}\Lambda'\Big\{\ln\left[(1+\eta_{n+1}^{(q)})(1+\eta_{n-1}^{(q)})\right] \\ & + \delta_{n,1}\ln(1+\exp(-\epsilon_{q-1}/T))\Big\}\int\frac{\mathrm{d}\omega}{2\pi}\mathrm{e}^{\mathrm{i}(\Lambda-\Lambda')\omega}\,\hat{G}_{m,q}(\omega).\end{aligned} \qquad (2.18)$$

2.4.3. If now $\rho^{(l)}(\xi)$ for $l \leqslant 2S-1$, $\rho_{\rm h}^{(2S)}(\xi)$ and the $\sigma_{\rm nh}^{(l)}(\Lambda)$ are the independent functions, then (2.15) can be rewritten as

$$\epsilon_{2S}(\xi) = -2 - \frac{1}{N}\sum_{q=0}^{2S} A_q + \int \mathrm{d}\omega\, \mathrm{e}^{-\mathrm{i}\xi\omega-\frac{1}{2}|\omega|}\, \frac{\sinh(\frac{\omega}{2})}{\sinh(\frac{\omega}{2}N)} \\ +T\sum_{q=0}^{2S-1}\int \mathrm{d}\xi' \ln\left(1+\exp\left(\frac{-\epsilon_q}{T}\right)\right)\int \frac{\mathrm{d}\omega}{2\pi}\mathrm{e}^{-\mathrm{i}(\xi-\xi')\omega}\frac{\sinh(\frac{\omega}{2}(q+1))}{\sinh(\frac{\omega}{2}N)} \\ +T\int \mathrm{d}\xi' \ln\left(1+\exp\left(\frac{\epsilon_{2S}}{T}\right)\right)\int \frac{\mathrm{d}\omega}{2\pi}\mathrm{e}^{-\mathrm{i}(\xi-\xi')\omega-\frac{1}{2}|\omega|}\,\frac{\sinh(\omega S)}{\sinh(\frac{\omega}{2}N)} \tag{2.19}$$

$$\ln(1+\exp(\epsilon_m/T)) = \frac{1}{T}\int \mathrm{d}\omega\, \mathrm{e}^{-\mathrm{i}\xi\omega}\, \frac{\sinh(\frac{\omega}{2}(2S-m))}{\sinh(\frac{\omega}{2}N)} \\ +\int \mathrm{d}\xi' \ln\left(1+\exp\left(\frac{\epsilon_{2S}}{T}\right)\right)\int \frac{\mathrm{d}\omega}{2\pi}\,\mathrm{e}^{-\mathrm{i}(\xi-\xi')\omega}\,\frac{\sinh(\frac{\omega}{2}(m+1))}{\sinh(\frac{\omega}{2}N)} \\ +\sum_{q=0}^{2S-1}\int \mathrm{d}\xi' \ln\left(1+\exp\left(\frac{-\epsilon_q}{T}\right)\right) \\ \times\int \frac{\mathrm{d}\omega}{2\pi}\mathrm{e}^{-\mathrm{i}(\xi-\xi')\omega}\, 2\cosh\left(\frac{\omega}{2}\right)\hat{G}_{m+1,q+1}(\omega) \\ -\sum_{q=0}^{2S-1}\int \mathrm{d}\Lambda' \ln\left(1+\eta_1^{(q+1)}\right)\int \frac{\mathrm{d}\omega}{2\pi}\mathrm{e}^{-\mathrm{i}(\xi-\Lambda')\omega}\hat{G}_{m+1,q+1}(\omega). \tag{2.20}$$

Equation (2.20) holds for $m < 2S$.

We have denoted

$$G_0(\xi) = \int \frac{\mathrm{d}\omega}{2\pi}\mathrm{e}^{\mathrm{i}\xi\omega}\left(2\cosh\left(\frac{\omega}{2}\right)\right)^{-1} \tag{2.21}$$

and

$$\hat{G}_{l,q}(\omega) = \frac{\sinh(\frac{\omega}{2}\min(q,l))\sinh[\frac{\omega}{2}(N-\max(q,l))]}{\sinh(\frac{\omega}{2}N)\sinh(\frac{\omega}{2})}. \tag{2.22}$$

Here $A = \frac{1}{N}\sum_{l=0}^{2S} A_l$ is the chemical potential.

In order to be completely defined, equations (2.17) and (2.18) require asymptotic conditions for the $\varphi_n^{(l)}$ as n tends to infinity. These boundary conditions are determined by the splitting scheme of the $(2S+1)$-fold multiplet, i.e. by the Zeeman and the crystal field energies through the Lagrange multipliers A_l for the conservation of the number of particles of a given colour. Note that $A_l - A$ is independent of the chemical potential. From (2.16) we obtain

$$\lim_{n\to\infty}\frac{1}{n}\varphi_n^{(l)} = A_{2S+1-l} - A_{2S-l} \geqslant 0 \tag{2.23}$$

in particular for pure Zeeman splitting it follows from the definition of the magnetization that $A_{2S+1-l} - A_{2S-l} = H$ for $l = 1, \ldots, 2S$.

The free energy of the system is given by

$$\frac{F}{N_{\mathrm{a}}} = -T \sum_{m=0}^{2S} \int \mathrm{d}\xi \, \ln(1+\exp(-\epsilon_m/T)) \, \frac{1}{\pi} \frac{\frac{1}{2}(m+1)}{\xi^2+\frac{1}{4}(m+1)^2} \tag{2.24}$$

or equivalently by

$$\begin{aligned} \frac{F}{N_{\mathrm{a}}} = & \int \mathrm{d}\omega \mathrm{e}^{-(S+1)|\omega|} \frac{\sinh(\frac{\omega}{2})}{\sinh(\frac{\omega}{2}N)} - 2 - \frac{1}{N}\sum_{l=0}^{2S} A_l - T\int \mathrm{d}\xi \, \ln(1+\exp(\epsilon_{2S}/T)) \\ & \times \int \frac{\mathrm{d}\omega}{2\pi} \exp\left(-\mathrm{i}\xi\omega - \frac{1}{2}|\omega|\right) \frac{\sinh(\frac{\omega}{2})}{\sinh(\frac{\omega}{2}N)} \\ & - T \sum_{m=0}^{2S-1} \int \mathrm{d}\xi \, \ln(1+\exp(-\epsilon_m/T)) \\ & \times \int \frac{\mathrm{d}\omega}{2\pi} \mathrm{e}^{-\mathrm{i}\xi\omega} \frac{\sinh[\frac{\omega}{2}(2S-m)]}{\sinh(\frac{\omega}{2}N)}. \end{aligned} \tag{2.25}$$

2.5. Relation between potentials and density functions

In order to derive the density functions for particles and holes from the thermodynamic potentials we modify the driving terms in equations (2.15) by replacing

$$2\frac{\frac{1}{2}(m+1)}{\xi^2+\frac{1}{4}(m+1)^2} \rightarrow 2x\frac{\frac{1}{2}(m+1)}{\xi^2+\frac{1}{4}(m+1)^2}.$$

Differentiating (2.15) and (2.16) with respect to x, we obtain by comparison with equations (2.6) and (2.7) that

$$\begin{aligned} \rho^{(m)}(\xi) &= \frac{1}{2\pi}\frac{\partial \epsilon_m}{\partial x}\left(1+\mathrm{e}^{\epsilon_m/T}\right)^{-1} \\ \rho_{\mathrm{h}}^{(m)}(\xi) &= \frac{1}{2\pi}\frac{\partial \epsilon_m}{\partial x}\left(1+\mathrm{e}^{-\epsilon_m/T}\right)^{-1} \\ \sigma_n^{(m)}(\Lambda) &= -\frac{1}{2\pi}\frac{\partial \varphi_n^{(m)}}{\partial x}\left(1+\eta_n^{(m)}\right)^{-1} \\ \sigma_{n\mathrm{h}}^{(m)}(\Lambda) &= -\frac{1}{2\pi}\frac{\partial \varphi_n^{(m)}}{\partial x}\left[1+\left(\eta_n^{(m)}\right)^{-1}\right]^{-1}. \end{aligned} \tag{2.26}$$

3. Special cases

In this section we discuss several limiting situations of the thermodynamic Bethe *ansatz* equations.

3.1. Spin $S = 1/2$

For $S = 1/2$ we recover the results presented in [6] if we identify ϵ_{2S} with Ψ and ϵ_0 with ϵ. Since there is only one spin degree of freedom the superscript m in $\varphi_n^{(m)}$ can be dropped. Note that for $S = 1/2$ the free energy can be brought into the simple form

$$F/N_\mathrm{a} = \Psi(0) - 2 - 2A.$$

3.2. Zero-temperature limit

Assuming that $A_{2S+1-l} \geqslant A_{2S-l}$ for $l = 1, \ldots, 2S$ it follows from the set of equations (2.16) that $\varphi_n^{(m)} > 0$ for all n and m, and all values of Λ. Consequently as $T \to 0$, $\sigma_n^{(m)}(\Lambda) \equiv 0$, so the strings corresponding to bound spin states are not occupied. The functions $\epsilon_m(\xi)$, on the other hand, may change sign as a function of ξ. We denote with $\epsilon_m^+(\xi)$ and $\epsilon_m^-(\xi)$ the positive and negative parts of $\epsilon_m(\xi)$, so that $\epsilon_m(\xi) = \epsilon_m^+(\xi) + \epsilon_m^-(\xi)$. Equations (2.15) then take the form

$$\begin{aligned}\epsilon_m^+(\xi) + \epsilon_m^-(\xi) = 2\frac{\frac{1}{2}(m+1)}{\xi^2 + \frac{1}{4}(m+1)^2} - 2(m+1) - \sum_{q=2S-m}^{2S} A_q - \sum_{q=0}^{2S} \int \mathrm{d}\xi' \, \epsilon_q^-(\xi') \\ \times \int \frac{\mathrm{d}\omega}{2\pi} \exp\left(\mathrm{i}(\xi - \xi')\omega - \frac{|\omega|}{2}(m + q - p_{m,q})\right) \\ \times \frac{\sinh\left[\frac{\omega}{2}(p_{m,q} + 1)\right]}{\sinh\left(\frac{\omega}{2}\right)}.\end{aligned} \tag{3.1}$$

The $\epsilon_m(\xi)$ are symmetric and monotonically decreasing functions of $|\xi|$ with zeroes at $\pm B_m$, i.e. $\epsilon_m(\pm B_m) = 0$. The function $\epsilon_m^+(\xi)$ is then non-vanishing in the interval $[-B_m, B_m]$ and identically zero elsewhere, while $\epsilon_m^-(\xi)$ is the complementary function.

The ground-state energy is obtained from equations (2.24) and (2.25)

$$\begin{aligned}\frac{E}{N_\mathrm{a}} = \sum_{m=0}^{2S} \int \mathrm{d}\xi \, \epsilon_m^-(\xi) \frac{1}{\pi} \frac{\frac{1}{2}(m+1)}{\xi^2 + \frac{1}{4}(m+1)^2} = \int \mathrm{d}\omega \, \mathrm{e}^{-(S+1)|\omega|} \frac{\sinh\left(\frac{\omega}{2}\right)}{\sinh\left(\frac{\omega}{2}N\right)} \\ - 2 - \frac{1}{N}\sum_{l=0}^{2S} A_l - \int \mathrm{d}\xi \, \epsilon_{2S}^+(\xi) \int \frac{\mathrm{d}\omega}{2\pi} \exp\left(-\mathrm{i}\xi\omega - \frac{1}{2}|\omega|\right) \frac{\sinh\left(\frac{\omega}{2}\right)}{\sinh\left(\frac{\omega}{2}N\right)} \\ + \sum_{m=0}^{2S-1} \int \mathrm{d}\xi \, \epsilon_m^-(\xi) \int \frac{\mathrm{d}\omega}{2\pi} \mathrm{e}^{-\mathrm{i}\xi\omega} \frac{\sinh\left(\frac{\omega}{2}(2S - m)\right)}{\sinh\left(\frac{\omega}{2}N\right)}.\end{aligned} \tag{3.2}$$

Equations (3.1) are coupled integral equations of the Fredholm type and have, in general, to be solved numerically. The most appropriate procedure is to fix the integration limits $\{B_m\}$ and determine the $\{A_m\}$ so that $\epsilon_m(\pm B_m) = 0$. Below we discuss some limiting cases in more detail.

3.3. Zero-temperature zero-field limit

In the absence of external fields $\epsilon_m^- \equiv 0$ for $m < 2S$ and only string states corresponding to charge–spin bound states of order $N = 2S + 1$ are occupied. Equation (3.1) then reduces to one integral equation for ϵ_{2S}

$$\epsilon_{2S}^+(\xi) + \left[\int_Q^\infty + \int_{-\infty}^{-Q}\right] d\xi' \, \epsilon_{2S}^-(\xi') \int \frac{d\omega}{2\pi} \exp(i(\xi - \xi')\omega) \frac{1 - \exp(-|\omega| N)}{1 - \exp(-|\omega|)}$$
$$= 2\frac{N/2}{\xi^2 + N^2/4} - (2 + A)N \tag{3.3}$$

where $Q = B_{2S}$ plays the role of the Fermi momentum. If $Q \to \infty$ the band is empty and we readily obtain

$$\epsilon_{2S}^+(\xi) = 2\frac{N/2}{\xi^2 + N^2/4} - (2 + A)N. \tag{3.4}$$

The condition for an empty band is of course $A \leqslant -2$.

When A is only slightly above -2 the system has a low electron density. Consequently Q is large but finite. Under these circumstances the integral equation can be solved iteratively [6,10] by reducing it to a sequence of Wiener–Hopf equations. After a lengthy calculation we obtain

$$Q = (2 + A)^{-1/2} \left(1 + (S/4\pi)(2 + A)^{1/2} \ln(2 + A) + \cdots\right)$$

and, to leading order, the number of electrons and the energy are given by

$$N_e/N_a = (N/\pi)(2 + A)^{1/2} \tag{3.5}$$
$$E/N_a = -(2N/3\pi)(2 + A)^{3/2} \tag{3.6}$$

as expected from a (one-dimensional) free electron density of states.

The situation of a full or almost full band can also be treated analytically. No holes corresponds to $Q = 0$, so that $\epsilon_{2S}^+ \equiv 0$. It is then straightforward to obtain $\epsilon_{2S}^-(\xi)$

$$\epsilon_{2S}^-(\xi) = \frac{2}{N} \mathrm{Re}\left[\psi\left(\frac{1}{2} + \frac{1}{N} + i\frac{\xi}{N}\right) - \psi\left(\frac{1}{2} + i\frac{\xi}{N}\right)\right] - (2 + A) \tag{3.7}$$

and similarly

$$\rho^{(2S)}(\xi) = \frac{1}{\pi N} \mathrm{Re}\left[\psi\left(\frac{1}{2} + \frac{1}{N} + i\frac{\xi}{N}\right) - \psi\left(\frac{1}{2} + i\frac{\xi}{N}\right)\right]. \tag{3.8}$$

Here ψ is the digamma function. It is easy to verify that

$$\frac{N_e}{N_a} = N \int_{-\infty}^{\infty} d\xi \, \rho^{(2S)}(\xi) = 1$$

and that

$$E/N_a = (2/N)[\psi(1 + 1/N) - \psi(1)] - (2 + A). \tag{3.9}$$

If Q is small but finite the system has a low density of holes. It is convenient in this case to rewrite the integral equation as

$$\begin{aligned}\epsilon_{2S}^{-}(\xi) + \int_{-Q}^{Q} \mathrm{d}\xi'\, \epsilon_{2S}^{+}(\xi') \int \frac{\mathrm{d}\omega}{2\pi} \exp(\mathrm{i}(\xi-\xi')\omega) \frac{1-\exp(-|\omega|)}{1-\exp(-|\omega|N)} \\ = \frac{2}{N} \mathrm{Re}\left[\psi\left(\frac{1}{2}+\frac{1}{N}+\mathrm{i}\frac{\xi}{N}\right) - \psi\left(\frac{1}{2}+\mathrm{i}\frac{\xi}{N}\right)\right] - (2+A) \qquad (3.10)\end{aligned}$$

which can be solved iteratively for small Q [6]. We obtain for the number of electrons and the energy:

$$\frac{N_{\mathrm{e}}}{N_{\mathrm{a}}} = 1 - \frac{2Q}{\pi N}\left[\psi\left(\frac{1}{2}+\frac{1}{N}\right) - \psi\left(\frac{1}{2}\right)\right] \qquad (3.11)$$

$$\begin{aligned}\frac{E}{N_{\mathrm{a}}} = \frac{2}{N}\left[\psi\left(1+\frac{1}{N}\right) - \psi(1)\right] - (2+A) \\ - \left\{\frac{2}{N}\left[\psi\left(\frac{1}{2}+\frac{1}{N}\right) - \psi\left(\frac{1}{2}\right)\right] - (2+A)\right\}\left(1-\frac{N_{\mathrm{e}}}{N_{\mathrm{a}}}\right). \qquad (3.12)\end{aligned}$$

3.4. Zero-temperature and small Zeeman splitting

We now consider the effect of a small magnetic field on the system with arbitrary band filling. In a small magnetic field all B_l ($l = 0, \ldots, 2S-1$) are finite but large. The difference in population of Zeeman levels is proportional to H, while the feedback of the field on ϵ_{2S} is proportional to H^2 and can be neglected. In other words, we use the zero-field result for ϵ_{2S}. After some algebra the integral equations can be cast into the following form

$$\begin{aligned}\epsilon_{m}^{+}(\xi) + \sum_{q=0}^{2S-1}\left[\int_{B_q}^{\infty} + \int_{-\infty}^{-B_q}\right] \mathrm{d}\xi'\, \epsilon_{q}^{-}(\xi') \\ \times \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi} \exp\left(\mathrm{i}(\xi-\xi')\omega + \frac{|\omega|}{2}\right) \hat{G}_{m+1,q+1}(\omega) \qquad (3.13) \\ = \int_{-Q}^{Q} \mathrm{d}\xi'\, \epsilon_{2S}^{+}(\xi')\, F_m(\xi-\xi') + 2\pi F_{2S-m}(\xi) - (m+1)(2S-m)(H/2)\end{aligned}$$

where

$$\begin{aligned}F_m(\xi) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi} \mathrm{e}^{\mathrm{i}\xi\omega} \frac{\sinh[\frac{\omega}{2}(m+1)]}{\sinh(\frac{\omega}{2}N)} \\ = \frac{1}{N} \frac{\sin[\pi(m+1)/N]}{\cosh[2\pi\xi/N] + \cos[\pi(m+1)/N]}. \qquad (3.14)\end{aligned}$$

Note that ϵ_{2S} is now part of the driving term. For sufficiently small magnetic fields $B_l \gg Q$, so the right-hand side of (3.13) can be approximated by

$$\begin{aligned}\frac{2\pi}{N} \frac{\sin[\pi(m+1)/N]}{\cosh[2\pi\xi/N]}\left[1 + \frac{1}{2\pi}\int_{-Q}^{Q} \mathrm{d}\xi' \exp\left(\frac{2\pi\xi'}{N}\right) \epsilon_{2S}^{+}(\xi')\right] \\ - (m+1)(2S-m)(H/2). \qquad (3.15)\end{aligned}$$

The corresponding set of integral equations obeyed by the density functions is obtained by dropping the field-dependent driving term in (3.13) and (3.15), and by making the following replacements

$$\epsilon_m^+ \to 2\pi\rho_{\rm h}^{(m)} \qquad \epsilon_m^- \to 2\pi\rho^{(m)} \qquad \epsilon_{2S}^+ \to 2\pi\rho_{\rm h}^{(2S)}.$$

Using the definition of the magnetization it is possible to show that the zero-field and low-field magnetic susceptibilities of the system are given by

$$\chi(0) = \frac{1}{4\pi^2}\frac{N(N^2-1)}{6}\frac{1+\int_{-Q}^{Q}\mathrm{d}\xi'\exp(2\pi\xi'/N)\,\rho_{\rm h}^{(2S)}(\xi')}{1+\frac{1}{2\pi}\int_{-Q}^{Q}\mathrm{d}\xi'\exp(2\pi\xi'/N)\;\epsilon_{2S}^+(\xi')} \tag{3.16a}$$

$$\frac{\chi(H)}{\chi(0)} = \left[1+\left(\frac{1}{N|\ln H|}\right)+\left(\frac{1}{N|\ln H|}\right)^2\ln\left(\frac{1}{N|\ln H|}\right)+\ldots\right]. \tag{3.16b}$$

Hence, $\chi(0)$ depends on the band filling and the susceptibility grows with decreasing electron concentration. (Note that J differs by a factor of 2 with respect to the definition in [14] for the Heisenberg model.) In a small but finite field we obtain logarithmic singularities as a consequence of the interference between the two Fermi points (scattering across the Fermi surface) [6, 14, 15].

3.5. High-temperature limit

If the temperature is much larger than the bandwidth, i.e. $T \gg 2$, we can neglect the independent terms in the integral equations. In the absence of driving terms the energy potentials ϵ_m and φ_n^m do not depend on ξ and Λ. The integral equations can then be reduced to a set of algebraic equations. We have succeeded in obtaining a general solution of this algebraic set of equations only for $S = 1/2$. This solution for $S = 1/2$ can be found in [6]. For general S we encountered similar difficulties as previously for the degenerate Anderson impurity (see [13]).

3.6. Mapping on the SU(N)-invariant Heisenberg chain

The procedure outlined below is analogous to the one presented in [13] to obtain the Coqblin–Schrieffer limit from the solution of the Anderson model. We define the following set of functions

$$\Theta_1^{(l+1)}(\Lambda) = \exp(-\epsilon_l/T) \qquad \Theta_{n+1}^{(l+1)}(\Lambda) = \eta_n^{(l+1)} \qquad l = 0,\ldots,2S-1 \tag{3.17}$$

where $n = 1,\ldots$. Since spin and charge excitations decouple in 1D models, we can freeze the charge excitations and incorporate the charge variables into the driving term. In this way we obtain the following set of integral equations

$$\begin{aligned}\ln\left[1+\left(\Theta_n^{(m)}\right)^{-1}\right] &= D_m(\Lambda) - \sum_{q=1}^{2S}\int \mathrm{d}\Lambda'\ln\left[\left(1+\Theta_{n+1}^{(q)}\right)\left(1+\Theta_{n-1}^{(q)}\right)\right]\\ &\quad\times\int\frac{\mathrm{d}\omega}{2\pi}\,\mathrm{e}^{\mathrm{i}(\Lambda-\Lambda')\omega}\hat{G}_{m,q}(\omega) + \sum_{q=1}^{2S}\int\mathrm{d}\Lambda'\ln\left(1+\Theta_n^{(q)}\right)\\ &\quad\times\int\frac{\mathrm{d}\omega}{2\pi}\,\mathrm{e}^{\mathrm{i}(\Lambda-\Lambda')\omega}\,2\cosh\left(\frac{\omega}{2}\right)\hat{G}_{m,q}(\omega).\end{aligned} \tag{3.18}$$

Here $D_m(\Lambda)$ are the driving terms, which are given by

$$D_m(\Lambda) = \frac{2\pi}{T} F_{2S-m}(\Lambda) + \int d\Lambda' \ln(1 + \exp(\epsilon_{2S}/T)) F_{m-1}(\Lambda) \tag{3.19}$$

where $m = 1, \ldots, 2S$.

For low-energy spin excitations the two driving terms have the same Λ-dependence. On the other hand, for a full band the driving term involving ϵ_{2S} vanishes and equations (3.18) and (3.19) reduce to the thermodynamic Bethe *ansatz* equations for the SU(N) Heisenberg chain. In this limit the charges are localized and have no dynamics.

4. Concluding remarks

We considered the N-fold-degenerate t–J model, which is completely integrable at the supersymmetric point $t = J = -V$ [10]. The components may be thought of as arising from combined spin and orbital degrees of freedom. Note that the model can be mapped onto a $(N + 1)$-component quantum lattice gas as introduced by Sutherland [5]. The additional degree of freedom corresponds to the charges. Using the string hypothesis we classified all the eigenstates of the Hamiltonian and derived the thermodynamic Bethe *ansatz* equations, generalizing in this way our results for the traditional ($S = 1/2$) t–J model. The procedure followed is in close analogy to the one used to derive the thermodynamics of the degenerate Anderson model for a magnetic impurity (in the limit $U \to \infty$) [13].

We have analysed the thermodynamic Bethe *ansatz* equations in several limits. For a full band the charges do not have dynamics and the equations reduce to those of the SU(N)-invariant Heisenberg chain. The properties of the model are particularly interesting at low T. The zero-field susceptibility is a decreasing function of the electron density. χ is expected to diverge as $N_e/N_a \to 0$ (1D van Hove singularity). In a finite but small magnetic field the susceptibility shows logarithmic singularities, which are characteristic of one-dimensional systems with SU(N) symmetry and arise due to the interference of the two Fermi-surface points [14,15]. The specific heat at low T is proportional to T. As a consequence of the logarithmic field singularitites the γ-coefficient is singular, in the sense that the limits $T \to 0$ and $H \to 0$ cannot be interchanged [16].

Although we classified the complex spin and charge rapidities (2.4) as bound states of electrons with different spin components, they are not bound states in the real sense, since no actual binding energy is involved (the binding energy is zero). The excitation spectrum is then similar to the one expected for a Fermi liquid or a marginal Fermi liquid. A ground-state crossover is believed to take place at the supersymmetric point of the t–J model (at least for low density of electrons) [9]. $t = J = -V$ is the boundary between normal Fermi liquid behaviour and a state with Cooper-pair-type bound states with finite pairing energy.

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